On hitting times of the winding processes of planar Brownian motion and of Ornstein-Uhlenbeck processes, via Bougerol's identity.

Stavros Vakeroudis

On hitting times of the winding processes of planar Brownian motion and of Ornstein-Uhlenbeck processes, via Bougerol's identity.

S. Vakeroudis *†

May 31, 2011

Abstract

Some identities in law in terms of planar complex valued Ornstein-Uhlenbeck processes $(Z_t = X_t + iY_t, t \geq 0)$ including planar Brownian motion are established and shown to be equivalent to the well known Bougerol identity for linear Brownian motion $(\beta_t, t \geq 0)$: for any fixed $u > 0$:

$$\sinh(\beta_u) \stackrel{(law)}{=} \hat{\beta}_{(\int_0^u ds \exp(2\beta_s))},$$

with $(\hat{\beta}_t, t \geq 0)$ a Brownian motion, independent of β.
These identities in law for 2-dimensional processes allow to study the distributions of hitting times $T_c^\theta \equiv \inf\{t : \theta_t = c\},\ (c > 0),\ \ T_{-d,c}^\theta \equiv \inf\{t : \theta_t \notin (-d, c)\},\ (c, d > 0)$ and more specifically of $T_{-c,c}^\theta \equiv \inf\{t : \theta_t \notin (-c, c)\},\ (c > 0)$ of the continuous winding processes $\theta_t = \mathrm{Im}(\int_0^t \frac{dZ_s}{Z_s}), t \geq 0$ of complex valued Ornstein-Uhlenbeck processes.

Key words: Planar Brownian motion, Ornstein-Uhlenbeck process, winding process, Bougerol's identity, exit time from a cone.

*Laboratoire de Probabilités et Modèles Aléatoires (LPMA) CNRS : UMR7599, Université Pierre et Marie Curie - Paris VI, 4 Place Jussieu, 75252 Paris Cedex 05, France. E-mail: stavros.vakeroudis@etu.upmc.fr

†Département de Mathématiques et de Biologie CNRS : UMR8544, Ecole Normale Supérieure, 46 rue d'Ulm, 75005 Paris, France.

1 Introduction

The conformal invariance of planar Brownian motion has deep consequences as to the structure of its trajectories (see, e.g., Le Gall [21]). In particular, a number of articles have been devoted to the study of its continuous winding process $(\theta_t, t \geq 0)$: Spitzer [33], Williams [35], Durrett [16], Messulam-Yor [28], Pitman-Yor [30], Le Gall-Yor [22], Bertoin-Werner [5], Yor [38], Pap-Yor [29], Bentkus-Pap-Yor [4]. In this paper, we take up again the study of the first hitting times:

$$T^{\theta}_{-d,c} \equiv \inf\{t : \theta_t \notin (-d, c)\}, \ (c, d > 0),$$

this time in relation with Bougerol's well-known identity (see Bougerol [8], Alili-Dufresne-Yor [2] and Yor [39]): for fixed $u > 0$:

$$\sinh(\beta_u) \overset{(law)}{=} \hat{\beta}_{\left(\int_0^u ds \exp(2\beta_s)\right)} \ ,$$

where $(\hat{\beta}_t, t \geq 0)$ is a Brownian motion[‡], independent of β.
In particular, it turns out that: for fixed $c > 0$:

$$\theta_{T^{\hat{\beta}}_c} \overset{(law)}{=} C_{a(c)}, \qquad (\star)$$

where $\hat{\beta}$ is a BM[‡] independent of $(\theta_u, u \geq 0)$, $T^{\hat{\beta}}_c = \inf\{t : \hat{\beta}_t = c\}$, $(C_t, t \geq 0)$ is a standard Cauchy process and $a(c) = \arg\sinh(c) \equiv \log\left(c + \sqrt{1 + c^2}\right), \ c \in \mathbb{R}$.
The identity $(\star)$ yields yet another proof of the celebrated Spitzer theorem:

$$\frac{2}{\log t}\, \theta_t \underset{t\to\infty}{\overset{(law)}{\longrightarrow}} C_1 \ ,$$

with the help of Williams' "pinching method" (see Williams [35] and Messulam-Yor [28]).

Moreover, we study the distributions of $T^{\theta}_{-\infty,c}$ and $T^{\theta}_{-c,c}$. In particular, we give explicit formulae for the density function of $T^{\theta}_{-c,c}$ and for the first moment of $\ln\left(T^{\theta}_{-c,c}\right)$.

‡When we simply write: Brownian motion, we always mean real-valued Brownian motion, starting from 0. For 2-dimensional Brownian motion, we indicate planar or complex BM.

The last section of the paper is devoted to developing similar results when planar Brownian motion is replaced by a complex valued Ornstein-Uhlenbeck process. We note that Bertoin-Werner [5] already made discussions of windings for planar Brownian motion using arguments related to Ornstein-Uhlenbeck processes.

Firstly, we obtain some analogue of $(\star)$ when $T_c^{\hat{\beta}}$ is replaced by $T_c^{(\lambda)} = T_{-c,c}^{\theta^Z} = \inf\{t : |\theta_t^Z| = c\}$, the corresponding time for an Ornstein-Uhlenbeck process with parameter λ. Secondly, we identify the distribution of $T_c^{(\lambda)}$. More specifically, we derive the asymptotics of $E\left[T_c^{(\lambda)}\right]$ for λ large and for λ small.

2 The Brownian motion case

2.1 A reminder on planar Brownian motion

Let $(Z_t = X_t + iY_t, t \geq 0)$ denote a standard planar Brownian motion, starting from $x_0 + i0, x_0 > 0$, where $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ are two independent linear Brownian motions, starting respectively from x_0 and 0.

As is well known (see e.g. Itô-McKean [20]), since $x_0 \neq 0$, $(Z_t, t \geq 0)$ does not visit a.s. the point 0 but keeps winding around 0 infinitely often. In particular, the continuous winding process $\theta_t = \mathrm{Im}(\int_0^t \frac{dZ_s}{Z_s}), t \geq 0$ is well defined.

Furthermore, there is the skew product representation:

$$\log |Z_t| + i\theta_t \equiv \int_0^t \frac{dZ_s}{Z_s} = (\beta_u + i\gamma_u) \Big|_{u=H_t=\int_0^t \frac{ds}{|Z_s|^2}}, \tag{1}$$

where $(\beta_u + i\gamma_u, u \geq 0)$ is another planar Brownian motion starting from $\log x_0 + i0$. For a study of the Bessel clock H, see Yor [36].

Rewriting (1) as:

$$\log |Z_t| = \beta_{H_t}; \quad \theta_t = \gamma_{H_t}, \tag{2}$$

we easily obtain that the total σ-fields $\sigma\{|Z_t|, t \geq 0\}$ and $\sigma\{\beta_u, u \geq 0\}$ are identical, whereas $(\gamma_u, u \geq 0)$ is independent from $(|Z_t|, t \geq 0)$.

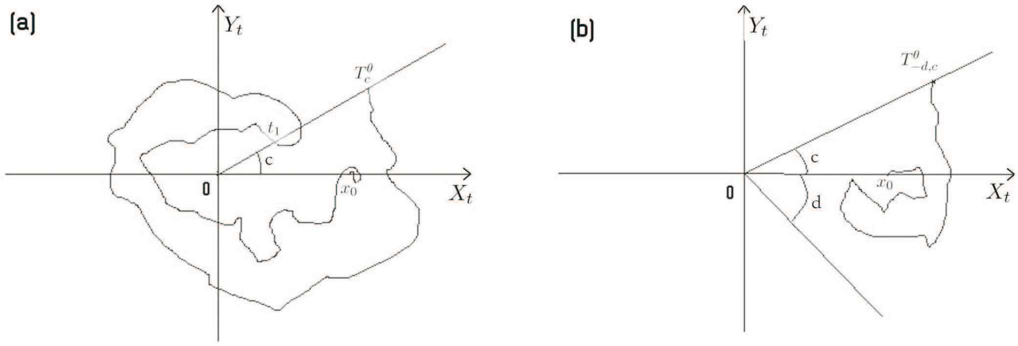

Figure 1: Exit times for a planar BM. This figure presents the exit times (a) T_c^θ (t_1 doesn't matter because the angle is negative) and (b) $T^\theta_{-d,c}$ for a planar BM starting from $x_0 + i0$.

A number of studies of the properties of the first hitting time (see Figure 1(b))

$$T^\theta_{-d,c} \equiv \inf\{t : \theta_t \notin (-d, c)\}, \quad (c, d > 0),$$

have been developed, going back to Spitzer [33].
In particular, it is well known (Spitzer [33], Burkholder [9], Revuz-Yor [32] Ex. 2.21/page 196) that:

$$E\left[(T^\theta_{-d,c})^p\right] < \infty \quad \text{if and only if} \quad p < \frac{\pi}{2(c+d)}. \tag{3}$$

Moreover, Spitzer's asymptotic theorem (see e.g. Spitzer [33]) states that:

$$\frac{2\theta_t}{\log t} \underset{t\to\infty}{\overset{(law)}{\longrightarrow}} C_1 \overset{(law)}{=} \gamma_{T_1^\beta}, \tag{4}$$

where C_1 is a standard Cauchy variable.

2.2 On the Laplace transform of the distribution of the hitting time $T_c^\theta \equiv T^\theta_{-\infty,c}$

Now, we use the representation (2) to access the distribution of T_c^θ (see Figure 1(a)). We define $T_c^\gamma \equiv \inf\{t : \gamma_t \notin (-\infty, c)\}$ the hitting time associated to

the Brownian motion $(\gamma_t, t \geq 0)$. Note that, from (2):
$H_{T_c^\theta} = T_c^\gamma$, hence: $T_c^\theta = H^{-1}_{u}\Big|_{u=T_c^\gamma}$, where

$$H_u^{-1} \equiv \inf\{t : H_t > u\} = \int_0^u ds \exp(2\beta_s) := A_u. \tag{5}$$

Thus, we have obtained:

$$T_c^\theta = A_{T_c^\gamma}, \tag{6}$$

where $(A_u, u \geq 0)$ and T_c^γ are independent, since β and γ are independent. We can write: $\beta_s = (\log x_0) + \beta_s^{(0)}$, with $(\beta_s^{(0)}, s \geq 0)$ a standard one-dimensional Brownian motion starting from 0. Then, we deduce from (6) that:

$$T_c^\theta = x_0^2 \left(\int_0^{T_c^\gamma} ds \exp(2\beta_s^{(0)}) \right). \tag{7}$$

From now on, for simplicity, we shall take $x_0 = 1$, but this is really no restriction, as the dependency in x_0, which is exhibited in (7), is very simple. We shall also make use of Bougerol's identity [8, 2] and [39] (p. 200), which is very useful to study the distribution of A_u (e.g. [26, 27]). For any fixed $u > 0$:

$$\sinh(\beta_u) \overset{(law)}{=} \hat{\beta}_{A_u} = \hat{\beta}_{(\int_0^u ds \exp(2\beta_s))}, \tag{8}$$

where on the right hand side, $(\hat{\beta}_t, t \geq 0)$ is a Brownian motion, independent of $A_u \equiv \int_0^u ds \exp(2\beta_s)$.
Thus, from (8) and (6), and as is well known [32], the law of $\beta_{T_c^\gamma}$ is the Cauchy law with parameter c, i.e., with density:

$$h_c(y) = \frac{c}{\pi(c^2 + y^2)},$$

we deduce that:

Proposition 2.1 *For fixed $c > 0$, there is the following identity in law:*

$$\sinh(C_c) \overset{(law)}{=} \hat{\beta}_{(T_c^\theta)}, \tag{9}$$

where, on the left hand side, $(C_c, c \geq 0)$ denotes a standard Cauchy process and on the right hand side, $(\hat{\beta}_u, u \geq 0)$ is a one-dimensional BM, independent from T_c^θ.

We may now identify the densities of the variables found on both sides of (9),i.e.:
on the left hand side: $\frac{1}{\sqrt{1+x^2}}h_c(\arg\sinh x) = \frac{1}{\sqrt{1+x^2}}h_c(a(x))$;
on the right hand side: $E\left[\frac{1}{\sqrt{2\pi T_c^\theta}}\exp\left(-\frac{x^2}{2T_c^\theta}\right)\right]$,
where $a(x) = \arg\sinh(x)$.
Thus, we have obtained the following:

Proposition 2.2 *The distribution of T_c^θ may be characterized by:*

$$E\left[\frac{1}{\sqrt{2\pi T_c^\theta}}\exp\left(-\frac{x}{2T_c^\theta}\right)\right] = \frac{1}{\sqrt{1+x}}\frac{c}{\pi(c^2+\log^2(\sqrt{x}+\sqrt{1+x}))}, \quad x \geq 0. \tag{10}$$

The proof of Proposition 2.2 follows from: $a(y) = \arg\sinh(y) \equiv \log(y + \sqrt{1+y^2})$ and by making the change of variable $y^2 = x$. Let us now define the probability:

$$Q_c = \sqrt{\frac{\pi c^2}{2T_c^\theta}} \cdot P \ .$$

The fact that Q_c is a probability follows from (10) by taking $x = 0$. Thus we obtain that $c\,E[\sqrt{\pi/2T_c^\theta}] = 1$, and we may write:

$$E_{Q_c}\left[\exp\left(-\frac{x}{2T_c^\theta}\right)\right] = \frac{1}{\sqrt{1+x}}\frac{1}{1+\frac{1}{c^2}\log^2(\sqrt{x}+\sqrt{1+x})}, \quad \forall x \geq 0, \tag{11}$$

which yields the Laplace transform of $1/T_c^\theta$ under Q_c.
Let us now take a look at what happens if we make $c \to \infty$. If we denote by $T_1^\beta \equiv \inf\{t : \beta_t = 1\}$ the first hitting time of level 1 for a standard BM β and by N a standard Gaussian variable $\mathcal{N}(0,1)$, from equation (11), we obtain:

$$\lim_{c\to\infty} E_{Q_c}\left[e^{-x/2T_c^\theta}\right] = E\left(e^{-xN^2/2}\right) = E\left(e^{-x/2T_1^\beta}\right), \tag{12}$$

which means that : $T_c^\theta \overset{(law)}{\underset{c\to\infty}{\longrightarrow}} T_1^\beta$. (At this point, one may wonder whether there is some kind of convergence in law involving $(\theta_u, u \geq 0)$, under Q_c, as $c \to \infty$, but, we shall not touch this point).

From Proposition 2.2 we deduce the following:

Corollary 2.3 *Let $\varphi(x)$ denote the Laplace transform (11), that is the Laplace transform of $1/2T_c^\theta$ under Q_c. Then, the Laplace transform of $1/2T_c^\theta$ under P is:*

$$E\left[\exp\left(-\frac{x}{2T_c^\theta}\right)\right] = \int_x^\infty \frac{dw}{\sqrt{w-x}}\,\varphi(w). \tag{13}$$

Proof of Corollary 2.3 From Fubini's theorem, we deduce from (11) that:

$$\begin{aligned}
E\left[\exp\left(-\frac{x}{2T_c^\theta}\right)\right] &= \int_0^\infty \frac{dy}{\sqrt{y}}\,E\left[\frac{1}{\sqrt{2\pi T_c^\theta}}\exp\left(-\frac{x+y}{2T_c^\theta}\right)\right] \\
&= \int_0^\infty \frac{dy}{\sqrt{y}}\,\varphi(x+y) \\
&\overset{y=xt}{=} \sqrt{x}\int_0^\infty \frac{dt}{\sqrt{t}}\,\varphi(x(1+t)) \\
&\overset{v=1+t}{=} \sqrt{x}\int_1^\infty \frac{dv}{\sqrt{v-1}}\,\varphi(xv) \\
&\overset{w=xv}{=} \int_x^\infty \frac{dw}{\sqrt{w-x}}\,\varphi(w),
\end{aligned}$$

which is formula (13).

$\square$

2.3 Some related identities in law

This subsection is strongly related to [15].
A slightly different look at the combination of Bougerol's identity (8) and the skew-product representation (1) lead to the following striking identities in law:

Proposition 2.4 *Let $(\delta_u, u \geq 0)$ be a 1-dimensional Brownian motion independent of the planar Brownian motion $(Z_u, u \geq 0)$, starting from $1 + i0$. Then, for any $b \geq 0$, the following identities in law hold:*

$$\text{(i) } H_{T_b^\delta} \overset{(law)}{=} T_{a(b)}^\beta \qquad \text{(ii) } \theta_{T_b^\delta} \overset{(law)}{=} C_{a(b)} \qquad \text{(iii) } \bar{\theta}_{T_b^\delta} \overset{(law)}{=} |C_{a(b)}|,$$

where C_A is a Cauchy variable with parameter A and $\bar{\theta}_u = \sup_{s\leq u}\theta_s$.

Proof of Proposition 2.4 From the symmetry principle (see [3] for the original Note and [17] for a detailed discussion), Bougerol's identity may be equivalently stated as:

$$\sinh(\bar{\beta}_u) \stackrel{(law)}{=} \bar{\delta}_{A_u(\beta)}. \tag{14}$$

Consequently, the laws of the first hitting times of a fixed level b by the processes on each side of (14) are identical, that is:

$$T^{\beta}_{a(b)} \stackrel{(law)}{=} H_{T^{\delta}_b},$$

which is (i).
(ii) follows from (i) since:

$$\theta_u \stackrel{(law)}{=} \gamma_{H_u},$$

with $(\gamma_s, s \geq 0)$ a Brownian motion independent of $(H_u, u \geq 0)$ and $(C_u, u \geq 0)$ may be represented as $(\gamma_{T^{\beta}_u}, u \geq 0)$.
(iii) follows from (ii), again with the help of the symmetry principle.

□

Remark 2.5 *Proposition 2.2 may also be derived from (iii) in Proposition 2.4. Indeed, for $c > 0$, starting from the LHS of (iii), and letting $N \sim \mathcal{N}(0,1)$ independent from T^{θ}_c:*

$$\begin{aligned} P\left(\bar{\theta}_{T^{\delta}_b} < c\right) &= P\left(T^{\delta}_b < T^{\theta}_c\right) = P\left(b < \bar{\delta}_{T^{\theta}_c}\right) \\ &= P\left(b < \sqrt{T^{\theta}_c}|N|\right) \\ &= P\left(\frac{b}{\sqrt{T^{\theta}_c}} < |N|\right) \\ &= \sqrt{\frac{2}{\pi}} E\left[\int_{b/\sqrt{T^{\theta}_c}}^{\infty} dy\, e^{-y^2/2}\right], \end{aligned} \tag{15}$$

while, on the RHS of (iii):

$$P\left(|C_{a(b)}| < c\right) = 2\int_0^c \frac{a(b)\, dy}{\pi(a^2(b) + y^2)} \stackrel{y=a(b)h}{=} \frac{2}{\pi}\int_0^{c/a(b)} \frac{dh}{1+h^2}. \tag{16}$$

Taking derivatives in (15) and (16) with respect to b and changing the variables $b = \sqrt{x}$, we obtain Proposition 2.2.

2.4 Recovering Spitzer's theorem

The identity (ii) in Proposition 2.4 is reminiscent of Williams' remark (see [35, 28]), that:

$$H_{T_r^R} \overset{(law)}{=} T^\delta_{\log r}, \tag{17}$$

where here R starts from 1 and δ starts from 0 (in fact, this is a consequence of (2)). For a number of variants of (17), see [37, 25]. This was D. Williams' starting point for a non-computational proof of Spitzer's result (4). We note that in (ii), T_b^δ is independent of the process $(\theta_u, u \geq 0)$ while in (17) T_r^R depends on $(\theta_u, u \geq 0)$. Actually, we can mimic Williams' "pinching method" to derive Spitzer's theorem (4) from (ii) in Proposition 2.4.

Proposition 2.6 ***(A new proof of Spitzer's theorem)***
As $t \to \infty$, $\theta_{T^\delta_{\sqrt{t}}} - \theta_t$ converges in law, which implies that:

$$\frac{1}{\log t}\left(\theta_{T^\delta_{\sqrt{t}}} - \theta_t\right) \xrightarrow[t\to\infty]{(P)} 0, \tag{18}$$

and, in turn, implies Spitzer's theorem (see formula (4)):

$$\frac{2}{\log t}\,\theta_t \xrightarrow[t\to\infty]{(law)} C_1.$$

Proof of Proposition 2.6 From equation (ii) of Proposition 2.4 we note:

$$\frac{1}{\log b}\,\theta_{T_b^\delta} \overset{(law)}{=} \frac{C_{a(b)}}{\log b} \xrightarrow[b\to\infty]{(law)} C_1.$$

So, for $b = \sqrt{t}$ we have:

$$\frac{2}{\log t}\,\theta_{T^\delta_{\sqrt{t}}} \xrightarrow[b\to\infty]{(law)} C_1.$$

On the other hand, following Williams' "pinching method", we note that:

$$\frac{1}{\log t}\left(\theta_{T^\delta_{\sqrt{t}}} - \theta_t\right) \xrightarrow[t\to\infty]{(law)} 0,$$

since $Z_u = x_0 + Z_u^{(0)}$ and also, as we change variables $u = tv$ and we use the scaling property, we obtain:

$$\theta_{T^\delta_{\sqrt{t}}} - \theta_t \equiv \mathrm{Im}\left(\int_t^{T^\delta_{\sqrt{t}}} \frac{dZ_u}{Z_u}\right) \xrightarrow[t\to\infty]{(law)} \mathrm{Im}\left(\int_1^{T_1^\delta} \frac{dZ_v^{(0)}}{Z_v^{(0)}}\right).$$

Here, the limit variable is -in our opinion- of no other interest than its existence which implies (18), hence (4).

□

2.5 On the distributions of $T_c^\theta \equiv T_{-\infty,c}^\theta$ and $T_{-c,c}^\theta$

Proposition 2.7 *The asymptotic equivalence:*

$$(\log t)\ P(T_c^\theta > t) \overset{t\to\infty}{\longrightarrow} (4c)/\pi\ , \tag{19}$$

holds.
As a consequence, for $\eta > 0$, $E[(\log T_c^\theta)_+^\eta] < \infty$ *if and only if* $\eta < 1$ *(where* $(\cdot)_+$ *denotes the positive part).*

Proof of Proposition 2.7 *a*) We rely upon the asymptotic distribution of $H_t \equiv \int_0^t \frac{ds}{|Z_s|^2}$ which is given by [32]:

$$\frac{4H_t}{(\log t)^2} \underset{t\to\infty}{\overset{(law)}{\longrightarrow}} T_1^\beta \equiv \inf\{t : \beta_t = 1\}, \tag{20}$$

or equivalently:

$$\frac{\log t}{2\sqrt{H_t}} \underset{t\to\infty}{\overset{(law)}{\longrightarrow}} |N|\,, \tag{21}$$

where N is a standard Gaussian variable $\mathcal{N}(0,1)$.
We note that, from the representation (2) of θ_t, the result (20) is equivalent to Spitzer's theorem [33]:

$$\frac{2\theta_t}{\log t} \underset{t\to\infty}{\overset{(law)}{\longrightarrow}} C_1 \overset{(law)}{=} \gamma_{T_1^\beta}, \tag{22}$$

where C_1 is a standard Cauchy variable.
b) We shall now use this, in order to deduce Proposition 2.7. We denote $S_t^\theta \equiv \sup_{s\leq t}\theta_s \equiv S_{H_t}^\gamma$ and we note that (from scaling):

$$P(T_c^\theta \geq t) = P(S_{H_t}^\gamma \leq c) = P(\sqrt{H_t}S_1^\gamma \leq c), \tag{23}$$

since γ and H are independent. Thus, we have (since $S_1^\gamma \stackrel{(law)}{=} |N|$ and by making the change of variable $x = \frac{cy}{\sqrt{H_t}}$):

$$\begin{aligned} P(T_c^\theta \geq t) &= \sqrt{\frac{2}{\pi}} E\left[\int_0^{c/\sqrt{H_t}} dx\, e^{-\frac{x^2}{2}}\right] \\ &= \sqrt{\frac{2}{\pi}}\; c\; E\left[\int_0^1 \frac{dy}{\sqrt{H_t}} \exp\left(-\frac{c^2y^2}{2H_t}\right)\right]. \end{aligned} \tag{24}$$

Thus, we now deduce from (21) that:

$$\frac{\log t}{2} P(T_c^\theta \geq t) \stackrel{t\to\infty}{\longrightarrow} \sqrt{\frac{2}{\pi}}\; c\; E\left[|N|\right] = \frac{2}{\pi}\; c. \tag{25}$$

which is precisely (19).
It is now elementary to deduce from (25) that: for $\eta > 0$:

$$E[(\log T_c^\theta)_+^\eta] < \infty \;\Leftrightarrow\; 0 < \eta < 1,$$

since (25) is equivalent to:

$$u\; P(\log T_c^\theta > u) \stackrel{u\to\infty}{\longrightarrow} \left(\frac{4c}{\pi}\right). \tag{26}$$

Consequently, Fubini's theorem yields:

$$E\left[(\log T_c^\theta)_+^\eta\right] = \int_0^\infty du\; \eta\; u^{\eta-1}\; P(\log T_c^\theta > u),$$

and from (26) this is finite if and only if:

$$\int_.^\infty du\; u^{\eta-2} < \infty \Leftrightarrow \eta < 1.$$

So, $E[(\log T_c^\theta)_+^\eta] < \infty \;\Leftrightarrow\; 0 < \eta < 1.$

$\square$

Now we give several examples of random times $T : C(\mathbb{R}_+, \mathbb{R}) \to \mathbb{R}_+$ which may be studied quite similarly to T_c^θ.
For such times T, it will always be true that: $H_{T(\theta)} = T(\gamma)$ is equivalent

to $T(\theta) = A_{T(\gamma)}$, defined with respect to Z, issued from $x_0 \neq 0$. Using Bougerol's identity, we obtain:

$$\sinh(\beta_{T(\gamma)}) \overset{(law)}{=} \hat{\beta}_{A_{T(\gamma)}} = \hat{\beta}_{(T(\theta))}. \tag{27}$$

where $(\hat{\beta}_u, u \geq 0)$ is a 1-dimensional Brownian motion independent of (β, γ) (or equivalently, of Z). Consequently, denoting by h_T the density of $\beta_{T(\gamma)}$, we deduce from (27) that:

$$E\left[\frac{1}{\sqrt{2\pi T(\theta)}} \exp(-\frac{x^2}{2T(\theta)})\right] = \frac{1}{\sqrt{1+x^2}}\ h_T(\log(x + \sqrt{1+x^2})), \tag{28}$$

or equivalently, changing x in $\sqrt{x}$, we obtain:

$$E\left[\frac{1}{\sqrt{2\pi T(\theta)}} \exp(-\frac{x}{2T(\theta)})\right] = \frac{1}{\sqrt{1+x}}\ h_T(\log(\sqrt{x} + \sqrt{1+x})). \tag{29}$$

In a number of cases, h_T is known explicitly, for example:
(i)

$$T(\gamma) = T^{\gamma}_{-d,c} \Leftrightarrow T(\theta) = \int_0^{T^{\gamma}_{-d,c}} ds\ \exp(2\beta_s) = T^{\theta}_{-d,c}.$$

Thus:

$$E\left[\frac{1}{\sqrt{2\pi T^{\theta}_{-d,c}}} \exp(-\frac{x}{2T^{\theta}_{-d,c}})\right] = \frac{1}{\sqrt{1+x}}\ h_{-d,c}(\log(\sqrt{x} + \sqrt{1+x})), \tag{30}$$

where $h_{-d,c}$ is the density of the variable $\beta_{T^{\gamma}_{-d,c}}$. The law of $\beta_{T^{\gamma}_{-d,c}}$ may be obtained from its characteristic function which is given by [32], page 73:

$$\begin{aligned} E\left[\exp(i\lambda\beta_{T^{\gamma}_{-d,c}})\right] &= E\left[\exp(-\frac{\lambda^2}{2}T^{\gamma}_{-d,c})\right] \\ &= \frac{\cosh(\frac{\lambda}{2}(c-d))}{\cosh(\frac{\lambda}{2}(c+d))}. \end{aligned}$$

In particular, for $c = d$, we recover the very classical formula:

$$E\left[\exp(i\lambda\beta_{T^{\gamma}_{-c,c}})\right] = \frac{1}{\cosh(\lambda c)}.$$

It is well known that [24, 6]:

$$
\begin{aligned}
E\left[\exp(i\lambda\beta_{T^{\gamma}_{-c,c}}))\right] &= \frac{1}{\cosh(\lambda c)} = \frac{1}{\cosh(\pi\lambda\frac{c}{\pi})} \\
&= \int_{-\infty}^{\infty} e^{i(\frac{\lambda c}{\pi})x}\frac{1}{2\pi}\frac{1}{\cosh(\frac{x}{2})}dx \\
&\overset{y=\frac{cx}{\pi}}{=} \int_{-\infty}^{\infty} e^{i\lambda y}\frac{1}{2\pi}\frac{\frac{\pi}{c}}{\cosh(\frac{y\pi}{2c})}dy \\
&= \int_{-\infty}^{\infty} e^{i\lambda y}\frac{1}{2c}\frac{1}{\cosh(\frac{y\pi}{2c})}dy. \qquad (31)
\end{aligned}
$$

Hence, the density of $\beta_{T^{\gamma}_{-c,c}}$ is:

$$
h_{-c,c}(x) = \left(\frac{1}{2c}\right)\frac{1}{\cosh(\frac{x\pi}{2c})} = \left(\frac{1}{c}\right)\frac{1}{e^{\frac{x\pi}{2c}} + e^{-\frac{x\pi}{2c}}},
$$

and

$$
h_{-c,c}\left(\log(\sqrt{x}+\sqrt{1+x})\right) = \left(\frac{1}{c}\right)\frac{1}{(\sqrt{x}+\sqrt{1+x})^{\zeta} + (\sqrt{x}+\sqrt{1+x})^{-\zeta}},
$$

where $\zeta = \frac{\pi}{2c}$. However using:

$$
(\sqrt{x}+\sqrt{1+x})^{-\zeta} = (\sqrt{1+x}-\sqrt{x})^{\zeta}, \qquad (32)
$$

we obtain:

$$
h_{-c,c}\left(\log(\sqrt{x}+\sqrt{1+x})\right) = \left(\frac{1}{c}\right)\frac{1}{(\sqrt{x}+\sqrt{1+x})^{\zeta} + (\sqrt{1+x}-\sqrt{x})^{\zeta}}. \qquad (33)
$$

So we deduce that (for $c = d$):

$$
\begin{aligned}
& E\left[\frac{1}{\sqrt{2\pi T^{\theta}_{-c,c}}}\exp(-\frac{x}{2T^{\theta}_{-c,c}})\right] \\
= & \left(\frac{1}{c}\right)\left(\frac{1}{\sqrt{1+x}}\right)\frac{1}{(\sqrt{x}+\sqrt{1+x})^{\zeta} + (\sqrt{1+x}-\sqrt{x})^{\zeta}}. \qquad (34)
\end{aligned}
$$

$\square$

(ii) As a second example of a random time T, let us consider the time introduced in [34], [10], exercise 6.2, p. 178 (we use a slightly different notation). Let $(\beta_t, t \geq 0)$ be a real valued Brownian motion and define, for $c > 0$:

$$\begin{aligned} T(\theta) &\equiv T_c^{\hat{\theta}} = \inf \left\{ t : \sup_{s \leq t} \theta_s - \inf_{s \leq t} \theta_s = c \right\}, \\ T(\gamma) &\equiv T_c^{\hat{\gamma}} = \inf \left\{ t : \sup_{s \leq t} \gamma_s - \inf_{s \leq t} \gamma_s = c \right\}. \end{aligned}$$

Thus, from the skew-product representation (1), $\theta_u \equiv \gamma_{H_u}$, by replacing $u = T_c^{\hat{\theta}}$, we obtain:

$$H_{T_c^{\hat{\theta}}} = T_c^{\hat{\gamma}} \Rightarrow T_c^{\hat{\theta}} = \int_0^{T_c^{\hat{\gamma}}} ds \ \exp\left(2\beta_s\right) \equiv A_{T_c^{\hat{\gamma}}}.$$

Thus:

$$E\left[\frac{1}{\sqrt{2\pi T_c^{\hat{\theta}}}} \exp(-\frac{x}{2T_c^{\hat{\theta}}}) \right] = \frac{1}{\sqrt{1+x}} \ h_c(\log(\sqrt{x} + \sqrt{1+x})), \tag{35}$$

where h_c is the density of the variable $\beta_{T_c^{\hat{\gamma}}}$. The law of $\beta_{T_c^{\hat{\gamma}}}$ may be obtained from its characteristic function which is given by [6, 10]:

$$\begin{aligned} E\left[\exp(i\lambda\beta_{T_c^{\hat{\gamma}}})\right] &= E\left[\exp(-\frac{\lambda^2}{2}T_c^{\hat{\gamma}})\right] = \frac{1}{(\cosh(\lambda\frac{c}{2}))^2} = \frac{1}{\left(\cosh(\pi\lambda\frac{c}{2\pi})\right)^2} \\ &= \int_{-\infty}^{\infty} e^{i\left(\frac{\lambda c}{2\pi}\right)x} \frac{1}{2\pi} \frac{x}{\sinh(\frac{x}{2})} dx \\ &\overset{y=\frac{cx}{2\pi}}{=} \int_{-\infty}^{\infty} e^{i\lambda y} \frac{1}{2\pi} \frac{\frac{2\pi y}{c}}{\sinh(\frac{\pi y}{c})} \frac{2\pi}{c} dy \\ &= \int_{-\infty}^{\infty} e^{i\lambda y} \frac{2\pi}{c^2} \frac{y}{\sinh(\frac{\pi y}{c})} dy. \end{aligned} \tag{36}$$

So, the density of $\beta_{T_c^{\hat{\gamma}}}$ is:

$$h_c(y) = \left(\frac{2\pi y}{c^2}\right) \frac{1}{\sinh(\frac{\pi y}{c})} = \frac{4\pi}{c^2} \frac{y}{e^{\frac{\pi y}{c}} - e^{-\frac{\pi y}{c}}},$$

and

$$h_c\left(\log(\sqrt{x}+\sqrt{1+x})\right) = \frac{4\pi}{c^2}\frac{\log(\sqrt{x}+\sqrt{1+x})}{(\sqrt{x}+\sqrt{1+x})^{\hat{\zeta}} - (\sqrt{x}+\sqrt{1+x})^{-\hat{\zeta}}},$$

where $\hat{\zeta} = \frac{\pi}{c}$. Thus:

$$\begin{aligned} & E\left[\frac{1}{\sqrt{2\pi T_c^{\hat{\theta}}}}\exp(-\frac{x}{2T_c^{\hat{\theta}}})\right] \\ &= \frac{4\pi}{c^2}\frac{1}{\sqrt{1+x}}\frac{\log(\sqrt{x}+\sqrt{1+x})}{(\sqrt{x}+\sqrt{1+x})^{\hat{\zeta}} - (\sqrt{1+x}-\sqrt{x})^{\hat{\zeta}}}. \end{aligned} \tag{37}$$

We note that this study may be related to [31]; and more precisely $\beta_{T_c^{\hat{\gamma}}}$ and $T_c^{\hat{\gamma}}$ correspond to the variables C_2 and $\hat{C}_2$ respectively (see e.g. Table 6 in p. 312).

□

Let us now return to the case of $T^{\theta}_{-c,c}$ (example (i)). More specifically, we shall obtain its density function $f(t)$.

Proposition 2.8 *The density function f of $T^{\theta}_{-c,c}$ is given by:*

$$f(t) = \frac{1}{\sqrt{2}c}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu_k)}{\Gamma(2\nu_k)}\frac{1}{\sqrt{t}}e^{-\frac{1}{4t}}M_{\frac{1}{2},\nu_k}(\frac{1}{2t}), \tag{38}$$

where $M_{a,b}(\cdot)$ is the Whittaker function with parameters a, b. Equivalently:

$$f(t) = \frac{\sqrt{2}}{c}\sum_{k=0}^{\infty}(-1)^k\frac{1}{\sqrt{t}}e^{-\frac{1}{2t}}\left(\frac{1}{2t}\right)^{\nu_k+\frac{1}{2}}\nu_k\sum_{n=0}^{\infty}\frac{\Gamma(\nu_k+n)}{\Gamma(2\nu_k+n+1)}\frac{1}{n!}\left(\frac{1}{2t}\right)^n, \tag{39}$$

where $\nu_k = \frac{\pi}{4c}(2k+1)$.

Proof of Proposition 2.8 The following calculation relies upon a private note by A. Comtet [11].We denote:

$$\varphi_\zeta(x) = (\sqrt{x}+\sqrt{1+x})^{\zeta} + (\sqrt{1+x}-\sqrt{x})^{\zeta}.$$

Noting:

$$\sqrt{1+x} = \cosh\frac{y}{2} \Longleftrightarrow y = 2\arg\cosh(\sqrt{1+x}), \tag{40}$$

we get:

$$\begin{aligned}
\varphi_\zeta(x) &= (\sinh\frac{y}{2} + \cosh\frac{y}{2})^\zeta + (\cosh\frac{y}{2} - \sinh\frac{y}{2})^\zeta \\
&= 2\cosh\frac{y\zeta}{2}.
\end{aligned}$$

Thus, from (34), we have:

$$II := E\left[\frac{1}{\sqrt{2\pi T^\theta_{-c,c}}}\exp(-\frac{x}{2T^\theta_{-c,c}})\right] = \frac{1}{\psi}\frac{1}{\cosh\frac{y}{2}}\frac{1}{\cosh\frac{\pi y}{2\psi}}, \tag{41}$$

where $\psi = 2c$. However, expanding $\cosh\frac{\pi y}{2\psi}$, we get:

$$\frac{1}{\cosh\frac{\pi y}{2\psi}} = 2\frac{e^{-\frac{\pi y}{2\psi}}}{1+e^{-\frac{\pi y}{\psi}}} = 2\sum_{k=0}^{\infty}\left(-e^{-\frac{\pi y}{\psi}}\right)^k e^{-\frac{\pi y}{2\psi}},$$

and from (41), we deduce that:

$$\begin{aligned}
II &= \sum_{k=0}^{\infty}\frac{2}{\psi}\frac{(-1)^k}{\cosh\frac{y}{2}}e^{-\frac{\pi}{2\psi}(2k+1)y} \\
&= \sum_{k=0}^{\infty}\frac{4(-1)^k}{\psi\sqrt{2}\sqrt{2\sinh\frac{y}{2}\cosh\frac{y}{2}}}\sqrt{\frac{\sinh\frac{y}{2}}{\cosh\frac{y}{2}}}e^{-\nu_k y} \\
&= \sum_{k=0}^{\infty}\frac{4(-1)^k}{\psi\sqrt{2}\sqrt{2\sinh\frac{y}{2}\cosh\frac{y}{2}}}\sqrt{\tanh\frac{y}{2}}e^{-\nu_k y},
\end{aligned}$$

where $\nu_k = \frac{\pi}{2\psi}(2k+1)$.
From (40), we have $1+x = \cosh^2\frac{y}{2} \Longleftrightarrow x = \sinh^2\frac{y}{2}$, thus:

$$(\tanh\frac{y}{2})^{1/2} = \sqrt{\frac{\sinh\frac{y}{2}}{\cosh\frac{y}{2}}} = \left(\frac{\sqrt{x}}{\sqrt{1+x}}\right)^{1/2} = \left(\frac{x}{1+x}\right)^{1/4}.$$

Moreover, we know that (see [1], equation 8.6.10, or [23]):

$$i\sqrt{\frac{\pi}{2\sinh y}}e^{-\nu_k y} = Q^{1/2}_{\nu_k-1/2}(\cosh y),$$

where $\{Q^a_b(\cdot)\}$ is the family of Legendre functions and $\cosh y = 2x+1$. So, we deduce:

$$II = \sum_{k=0}^{\infty} \frac{4(-i)}{\psi\sqrt{\pi}}(-1)^k \left(\frac{x}{1+x}\right)^{1/4} Q^{1/2}_{\nu_k-1/2}(2x+1). \tag{42}$$

By using formula 7.621.9, page 864 in [18]:

$$\int_0^\infty e^{-sw} M_{l,\nu_k}(w)\frac{dw}{w} = \frac{2\Gamma(1+2\nu_k)\, e^{-i\pi l}}{\Gamma(\frac{1}{2}+\nu_k+l)}\left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^{l/2} Q^l_{\nu_k-1/2}(2s), \tag{43}$$

with: $l=\frac{1}{2}$, $\nu_k = \frac{\pi}{2\psi}(2k+1)$, $s = x+\frac{1}{2}$ and $M_{\cdot,\cdot}(\cdot)$ denoting the Whittaker function, which is defined as:

$$M_{a,b}(w) = w^{b+\frac{1}{2}}e^{-\frac{1}{2}w}\frac{\Gamma(2b+1)}{\Gamma(\frac{1}{2}+b-a)}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{1}{2}+b-a+n)}{\Gamma(2b+1+n)}\frac{w^n}{n!}.$$

we have:

$$-2i\frac{\Gamma(1+2\nu_k)}{\Gamma(1+\nu_k)}\left(\frac{x}{1+x}\right)^{1/4} Q^{1/2}_{\nu_k-1/2}(2x+1) = \int_0^\infty e^{-sw}M_{1/2,\nu_k}(w)\frac{dw}{w}. \tag{44}$$

From (42) and by changing the variable $w=\frac{1}{2t}$, we deduce:

$$\begin{aligned} II &= \sum_{k=0}^{\infty} \frac{2}{\psi\sqrt{\pi}}(-1)^k\frac{\Gamma(\nu_k+1)}{\Gamma(2\nu_k+1)}\int_0^\infty \frac{dw}{w}\exp\left(-w(x+\frac{1}{2})\right)M_{1/2,\nu_k}(w) \\ &= \sum_{k=0}^{\infty}\int_0^\infty \frac{dt}{t}\frac{2}{\psi\sqrt{\pi}}(-1)^k\frac{\Gamma(\nu_k+1)}{\Gamma(2\nu_k+1)}\exp\left(-\frac{1}{4t}-\frac{x}{2t}\right)M_{1/2,\nu_k}(\frac{1}{2t}). \end{aligned} \tag{45}$$

By using the equations (41) and (45), we conclude:

$$
\begin{aligned}
& E\left[\frac{1}{\sqrt{2\pi T^{\theta}_{-c,c}}}\exp\left(-\frac{x}{2T^{\theta}_{-c,c}}\right)\right] \\
= & \sum_{k=0}^{\infty}\int_0^{\infty}\frac{dt}{t}\frac{2}{\psi\sqrt{\pi}}(-1)^k\frac{\Gamma(\nu_k+1)}{\Gamma(2\nu_k+1)}\exp\left(-\frac{1}{4t}-\frac{x}{2t}\right)M_{\frac{1}{2},\nu_k}(\frac{1}{2t}) \\
= & \sum_{k=0}^{\infty}\int_0^{\infty}\frac{dt}{t}\frac{2}{\psi\sqrt{\pi}}(-1)^k\frac{\Gamma(\frac{\pi}{4c}(2k+1)+1)}{\Gamma(2\frac{\pi}{4c}(2k+1)+1)}\exp\left(-\frac{1}{4t}-\frac{x}{2t}\right)M_{\frac{1}{2},\frac{\pi}{4c}(2k+1)}(\frac{1}{2t}).
\end{aligned}
\tag{46}
$$

Thus, the density function f of $T^{\theta}_{-c,c}$ is given by:

$$
f(t) = \frac{2\sqrt{2}}{\psi}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu_k+1)}{\Gamma(2\nu_k+1)}\frac{1}{\sqrt{t}}e^{-\frac{1}{4t}}M_{\frac{1}{2},\nu_k}(\frac{1}{2t}) \tag{47}
$$

$$
= \frac{\sqrt{2}}{c}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\frac{\pi}{4a}(2k+1)+1)}{\Gamma(\frac{\pi}{2a}(2k+1)+1)}\frac{1}{\sqrt{t}}e^{-\frac{1}{4t}}M_{\frac{1}{2},\frac{\pi}{4a}(2k+1)}(\frac{1}{2t}) \tag{48}
$$

$$
= \frac{\sqrt{2}}{c}\sum_{k=0}^{\infty}(-1)^k\frac{\nu_k\Gamma(\nu_k)}{2\nu_k\Gamma(2\nu_k)}\frac{1}{\sqrt{t}}e^{-\frac{1}{4t}}M_{\frac{1}{2},\nu_k}(\frac{1}{2t}), \tag{49}
$$

where the Whittaker function $M_{\frac{1}{2},\nu_k}(\frac{1}{2t})$ is:

$$
\begin{aligned}
& M_{\frac{1}{2},\frac{\pi}{4c}(2k+1)}(\frac{1}{2t}) \\
= & \left(\frac{1}{2t}\right)^{\frac{\pi}{4c}(2k+1)+\frac{1}{2}} e^{-\frac{1}{4t}}\frac{\Gamma(\frac{\pi}{2c}(2k+1)+1)}{\Gamma(\frac{\pi}{4c}(2k+1))}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{\pi}{4c}(2k+1)+n)}{\Gamma(\frac{\pi}{2c}(2k+1)+1+n)}\frac{1}{n!}\left(\frac{1}{2t}\right)^n \\
= & \left(\frac{1}{2t}\right)^{\nu_k+\frac{1}{2}} e^{-\frac{1}{4t}}\frac{\Gamma(2\nu_k+1)}{\Gamma(\nu_k)}\sum_{n=0}^{\infty}\frac{\Gamma(\nu_k+n)}{\Gamma(2\nu_k+1+n)}\frac{1}{n!}\left(\frac{1}{2t}\right)^n \\
= & \left(\frac{1}{2t}\right)^{\nu_k+\frac{1}{2}} e^{-\frac{1}{4t}}(2\nu_k)\frac{\Gamma(2\nu_k)}{\Gamma(\nu_k)}\sum_{n=0}^{\infty}\frac{\Gamma(\nu_k+n)}{(2\nu_k+n)\Gamma(2\nu_k+n)}\frac{1}{n!}\left(\frac{1}{2t}\right)^n.
\end{aligned}
\tag{50}
$$

Thus, from (49) and (50), we deduce (39).

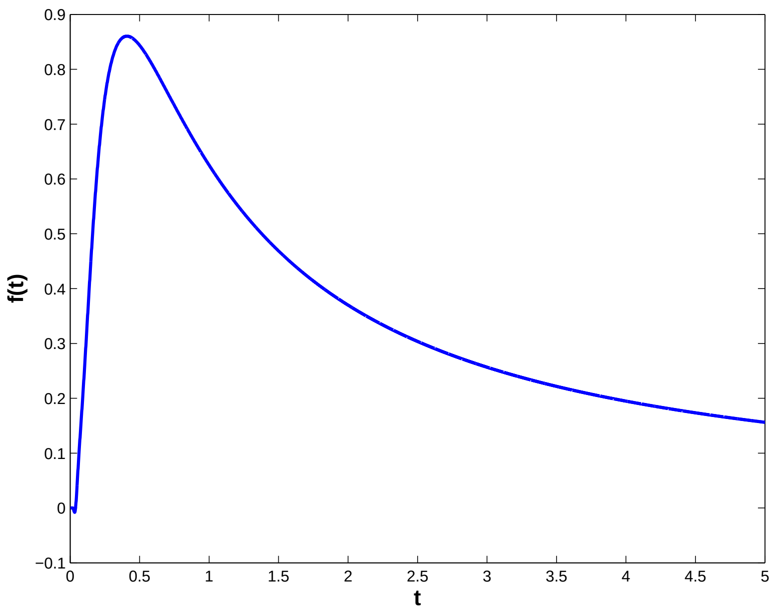

Figure 2: Graph of $f_{9,9}(t)$, with $c = 2\pi$.

$\square$

Next, we present the graphs of different approximations $f_{K,N}(t)$ of $f(t)$, in (39), where $f_{K,N}$ denotes the sum in the series in (39) of the terms for $k \leq K$, and $n \leq N$.

Remark 2.9

- *Figure 2 represents the approximation of the density function f with respect to the time t (for K and $N \leq 9$), with $c = 2\pi$, whereas Figure 3 represents the approximation of f with respect to the time t for several values of k and n, with $c = 2\pi$.*

- *From Figure 3, we may remark that the approximation K and $N \leq 9$ is sufficiently good (comparing to the one for K and $N \leq 100$).*

- *For the case K and $N \leq 9$ it seems that locally, in a small area around 0, $f(t) < 0$ which is not right. This is due to the first negative ($k = 1$) term of the sum and due to the fact that we have omitted many terms. However, this is not a problem because it appears only locally. Similar irregularities have already been observed in previous articles [19] p.275.*

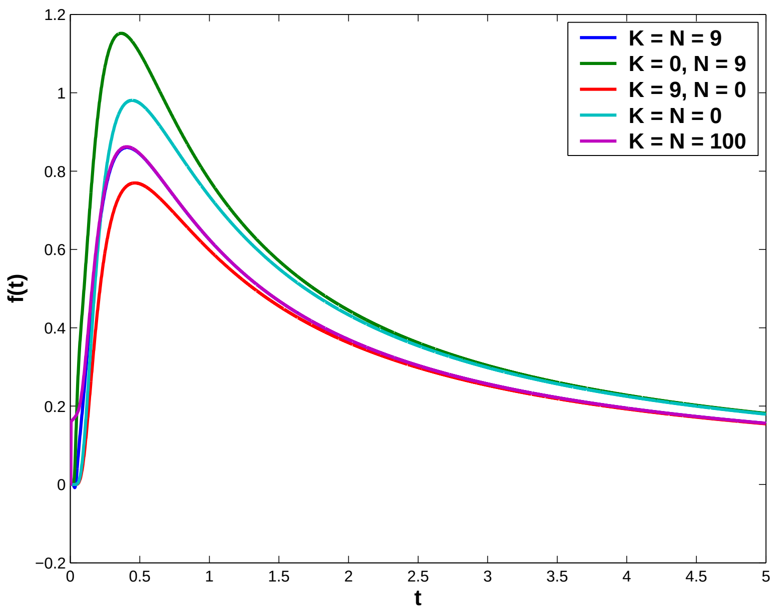

Figure 3: Graph of $f_{K,N}(t)$ for several values of K and N, with $c = 2\pi$.

2.6 On the first moment of $\ln\left(T^{\theta}_{-c,c}\right)$

This subsection is related to a result in [12].

Proposition 2.10 *The first moment of* $\ln\left(T^{\theta}_{-c,c}\right)$ *has the following integral representation:*

$$E\left[\ln\left(T^{\theta}_{-c,c}\right)\right] = 2\int_0^\infty \frac{dz}{\cosh\left(\frac{\pi z}{2}\right)} \ln\left(\sinh\left(cz\right)\right) + \ln\left(2\right) + c_E, \tag{51}$$

where $c_E = -\Gamma'(1)$ *is the Euler-Mascheroni constant (also called Euler's constant).*

Proof of Proposition 2.10 Let us return to equations (2) and (6). So, for $t = T^{\theta}_{-c,c}$, we have:

$$\theta_{T^{\theta}_{-c,c}} = \gamma_{H_{T^{\theta}_{-c,c}}} \Longleftrightarrow H_{T^{\theta}_{-c,c}} = T^{\gamma}_{-c,c} \Longleftrightarrow T^{\theta}_{-c,c} = A_{T^{\gamma}_{-c,c}}. \tag{52}$$

Thus, for $\varepsilon > 0$:

$$E\left[\left(T^{\theta}_{-c,c}\right)^{\varepsilon}\right] = E\left[\left(A_{T^{\gamma}_{-c,c}}\right)^{\varepsilon}\right].$$

Consider $(\delta_u, u \geq 0)$ a Brownian motion, independent of A_t. Then, Bougerol's identity and the scaling property yield ($\mathcal{G}_a$ denotes a gamma variable with parameter a, and $N^2 \overset{(law)}{=} 2\mathcal{G}_{1/2}$):

$$\begin{aligned} E\left[(\sinh(B_t))^{2\varepsilon}\right] &= E\left[(\delta_{A_t})^{2\varepsilon}\right] = E\left[A_t^{\varepsilon}(\delta_1)^{2\varepsilon}\right] \\ &= E\left[A_t^{\varepsilon}\right] E\left[\left(2\mathcal{G}_{1/2}\right)^{\varepsilon}\right] \\ &= E\left[A_t^{\varepsilon}\right] (2^{\varepsilon}) \frac{\Gamma\left(\frac{1}{2}+\varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)}, \end{aligned}$$

because

$$E\left[\left(\mathcal{G}_{1/2}\right)^{\varepsilon}\right] = \int_0^{\infty} x^{\varepsilon+\frac{1}{2}-1} \frac{e^{-x}}{\Gamma\left(\frac{1}{2}\right)} dx = \frac{\Gamma\left(\frac{1}{2}+\varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)}.$$

Thus, for $t = T^{\gamma}_{-c,c}$, we have:

$$E\left[\left(\sinh\left(B_{T^{\gamma}_{-c,c}}\right)\right)^{2\varepsilon}\right] = E\left[A^{\varepsilon}_{T^{\gamma}_{-c,c}}\right] (2^{\varepsilon}) \frac{\Gamma\left(\frac{1}{2}+\varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)}. \tag{53}$$

Recall that [24, 6]:

$$E\left[\exp(i\lambda B_{T^{\gamma}_{-c,c}}))\right] = E\left[\exp(-\frac{\lambda^2}{2} T^{\gamma}_{-c,c}))\right] = \frac{1}{\cosh(\lambda c)},$$

and the density of $\beta_{T^{\gamma}_{-c,c}}$ is:

$$h_{-c,c}(y) = \left(\frac{1}{2c}\right) \frac{1}{\cosh(\frac{y\pi}{2c})} = \left(\frac{1}{c}\right) \frac{1}{e^{\frac{y\pi}{2c}} + e^{-\frac{y\pi}{2c}}}.$$

Thus, on the left hand side of (53), we have:

$$\begin{aligned} E\left[\left(\sinh\left(B_{T^{\gamma}_{-c,c}}\right)\right)^{2\varepsilon}\right] &= \int_{-\infty}^{\infty} \frac{dy}{2c} \frac{1}{\cosh(\frac{\pi y}{2c})} (\sinh(y))^{2\varepsilon} \\ &= \int_0^{\infty} \frac{dy}{c} \frac{1}{\cosh(\frac{\pi y}{2c})} (\sinh y)^{2\varepsilon} \\ &= \int_0^{\infty} dz \frac{1}{\cosh(\frac{\pi z}{2})} (\sinh(cz))^{2\varepsilon}, \end{aligned}$$

where we have made the change of variable $z = \frac{y}{c}$. Hence, from (53), by writing:

$$E\left[A^{\varepsilon}_{T^{\gamma}_{-c,c}}\right] = E\left[\left(T^{\theta}_{-c,c}\right)^{\varepsilon}\right] = E\left[e^{\varepsilon\ \ln\left(T^{\theta}_{-c,c}\right)}\right],$$

we deduce:

$$\frac{\Gamma\left(\frac{1}{2}+\varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)} E\left[e^{\varepsilon\ \ln\left(T^{\theta}_{-c,c}\right)}\right] = \frac{1}{2^{\varepsilon}} \int_0^{\infty} \frac{dz}{\cosh(\frac{\pi z}{2})} \left(\sinh(cz)\right)^{2\varepsilon},$$

and by removing 1 from both sides, we obtain:

$$\frac{\Gamma\left(\frac{1}{2}+\varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)} E\left[e^{\varepsilon\ \ln\left(T^{\theta}_{-c,c}\right)}\right] - 1 = \int_0^{\infty} \frac{dz}{\cosh(\frac{\pi z}{2})} \left(\frac{\left(\sinh(cz)\right)^{2\varepsilon}}{2^{\varepsilon}} - 1\right). \tag{54}$$

On the left hand side, we apply the trivial identity $ab - 1 = a(b-1) + a - 1$ with $a = \frac{\Gamma\left(\frac{1}{2}+\varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)}$ and $b = E\left[e^{\varepsilon\ \ln\left(T^{\theta}_{-c,c}\right)}\right]$, we divide by ε and we take the limit for $\varepsilon \to 0$. Thus:

$$\begin{aligned} \frac{a(b-1)}{\varepsilon} &= \frac{\Gamma\left(\frac{1}{2}+\varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{E\left[e^{\varepsilon\ \ln\left(T^{\theta}_{-c,c}\right)}\right] - 1}{\varepsilon} \\ &\overset{\varepsilon \to 0}{\longrightarrow} E\left[\ln\left(T^{\theta}_{-c,c}\right)\right], \end{aligned}$$

and:

$$\begin{aligned} \frac{a-1}{\varepsilon} &= \frac{1}{\varepsilon}\left(\frac{\Gamma\left(\frac{1}{2}+\varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)} - 1\right) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{\Gamma\left(\frac{1}{2}+\varepsilon\right) - \Gamma\left(\frac{1}{2}\right)}{\varepsilon}\right) \\ &\overset{\varepsilon \to 0}{\longrightarrow} \frac{1}{\sqrt{\pi}}\Gamma'\left(\frac{1}{2}\right) = \frac{1}{\sqrt{\pi}}\left(-\sqrt{\pi}\right)\left(c_E + 2\ln 2\right) = -\left(c_E + 2\ln 2\right). \end{aligned}$$

On the right hand side of (54), we have:

$$\frac{1}{\varepsilon}\left[\left(\frac{\left(\sinh(cz)\right)^2}{2}\right)^{\varepsilon} - 1\right] = \frac{1}{\varepsilon}\left[\exp\left(\varepsilon\ \ln\left(\frac{\left(\sinh(cz)\right)^2}{2}\right)\right) - 1\right],$$

hence:

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^\infty \frac{dz}{\cosh(\frac{\pi z}{2})}\left(\frac{(\sinh(cz))^{2\varepsilon}}{2^\varepsilon}-1\right)\\
\xrightarrow{\varepsilon\to 0}\quad &\int_0^\infty \frac{dz}{\cosh(\frac{\pi z}{2})}\ln\left(\frac{(\sinh(cz))^{2}}{2}\right)\\
=\quad & -\ln(2)+2\int_0^\infty \frac{dz}{\cosh(\frac{\pi z}{2})}\left(\ln\left(\sinh(cz)\right)\right)\ ,
\end{aligned}$$

which finishes the proof.

□

Remark 2.11 *a) We denote now:*

$$F(c)\equiv\int_0^\infty \frac{dz}{\cosh\left(\frac{\pi z}{2}\right)}\ln\left(\sinh\left(cz\right)\right). \tag{55}$$

Thus:

$$F(c)-\ln(c)\equiv\int_0^\infty \frac{dz}{\cosh\left(\frac{\pi z}{2}\right)}\ln\left(\frac{\sinh\left(cz\right)}{c}\right)\xrightarrow{c\to 0}\int_0^\infty \frac{dz\ \ln(z)}{\cosh\left(\frac{\pi z}{2}\right)}\approx -0.7832. \tag{56}$$

b) More generally, we denote:

$$F(c,\delta)\equiv\int_0^\infty \frac{dz}{\cosh\left(\delta z\right)}\ln\left(\sinh\left(cz\right)\right), \tag{57}$$

and, changing the variables: $z=\frac{\pi}{2\delta}u$, *we obtain:*

$$F\left(c,\delta\right)=\left(\frac{\pi}{2\delta}\right)\int_0^\infty \frac{du}{\cosh\left(\frac{\pi}{2}u\right)}\ln\left(\sinh\left(c\frac{\pi}{2\delta}u\right)\right)=\frac{\pi}{2\delta}\ F\left(c\ \frac{\pi}{2\delta}\right). \tag{58}$$

3 The Ornstein-Uhlenbeck case

3.1 An identity in law for Ornstein-Uhlenbeck processes, which is connected to Bougerol's identity

Consider the complex valued Ornstein-Uhlenbeck (OU) process:

$$Z_t = z_0 + \tilde{Z}_t - \lambda \int_0^t Z_s ds, \tag{59}$$

where $\tilde{Z}_t$ is a complex valued Brownian motion (BM), $z_0 \in \mathbb{C}$ and $\lambda \geq 0$ and $T_c^{(\lambda)} \equiv T_{-c,c}^{\theta^Z} \equiv \inf\left\{t \geq 0 : \left|\theta_t^Z\right| = c\right\}$ (θ_t^Z is the continuous winding process associated to Z) denoting the first hitting time of the symmetric conic boundary of angle c for Z. It is well known that [32]:

$$\begin{aligned} Z_t &= e^{-\lambda t}\left(z_0 + \int_0^t e^{\lambda s} d\tilde{Z}_s\right) \\ &= e^{-\lambda t}\left(\mathbb{B}_{\alpha_t}\right), \end{aligned} \tag{60}$$

where, in the second equation, with the help of Dambis-Dubins-Schwarz Theorem, $(\mathbb{B}_t, t \geq 0)$ is a complex valued Brownian motion starting from z_0 and

$$\alpha_t = \int_0^t e^{2\lambda s} ds = \frac{e^{2\lambda t} - 1}{2\lambda} \quad .$$

We are interested in the study of the continuous winding process $\theta_t^Z = \mathrm{Im}(\int_0^t \frac{dZ_s}{Z_s}), t \geq 0$. By applying Itô's formula to (60), we have:

$$dZ_s = e^{-\lambda s}(-\lambda)\mathbb{B}_{\alpha_s} ds + e^{-\lambda s} d\left(\mathbb{B}_{\alpha_s}\right).$$

We divide by Z_s and we obtain:

$$\frac{dZ_s}{Z_s} = -\lambda \; ds + \frac{d\mathbb{B}_{\alpha_s}}{\mathbb{B}_{\alpha_s}},$$

hence:

$$\mathrm{Im}\left(\frac{dZ_s}{Z_s}\right) = \mathrm{Im}\left(\frac{d\mathbb{B}_{\alpha_s}}{\mathbb{B}_{\alpha_s}}\right),$$

which means that:

$$\theta_t^Z = \theta_{\alpha_t}^{\mathbb{B}} \quad .$$

Thus, the following holds:

Proposition 3.1 *Using the previously introduced notation, we have:*

$$\theta_t^Z = \theta_{\alpha_t}^{\mathbb{B}}, \tag{61}$$

and:

$$T_c^{(\lambda)} = \frac{1}{2\lambda} \ln \left(1 + 2\lambda T_{-c,c}^{\theta^{\mathbb{B}}}\right), \tag{62}$$

where $T_{-c,c}^{\theta^{\mathbb{B}}}$ is the exit time from a cone of angle c for the complex valued BM $\mathbb{B}$.

Proof of Proposition 3.1 We define

$$\begin{aligned} T_c^{(\lambda)} &\equiv T_{-c,c}^{\theta^Z} \equiv \inf \left\{t \geq 0 : \left|\theta_t^Z\right| = c\right\} \\ &= \inf \left\{t \geq 0 : \left|\theta_{\alpha_t}^{\mathbb{B}}\right| = c\right\}. \end{aligned} \tag{63}$$

Thus, we deduce that $\alpha_{T_c^{(\lambda)}} = T_{-c,c}^{\theta^{\mathbb{B}}} \equiv T_{-c,c}^{\theta}$. However, $T_{-c,c}^{\theta}$ (the exit time from a cone for the BM) has already been studied in the previous chapter and we know the explicit formula of its density function (Proposition 2.8). Thus:

$$T_c^{(\lambda)} = \alpha^{-1}\left(T_{-c,c}^{\theta^{\mathbb{B}}}\right) = \alpha^{-1}\left(T_{-c,c}^{\theta}\right), \tag{64}$$

where $\alpha^{-1}(t) = \frac{1}{2\lambda} \ln (1 + 2\lambda t)$. Consequently:

$$T_c^{(\lambda)} = \frac{1}{2\lambda} \ln \left(1 + 2\lambda T_{-c,c}^{\theta}\right),$$

and:

$$E\left[T_c^{(\lambda)}\right] = \frac{1}{2\lambda} E\left[\ln \left(1 + 2\lambda T_{-c,c}^{\theta}\right)\right], \tag{65}$$

which finishes the proof.

□

From now on, for simplicity, we shall take $z_0 = 1$ (but this is really no restriction, as the dependency in z_0, which is exhibited in (7), is very simple). The following Proposition may be considered as an extension of the identity in law (ii) in Proposition 2.4, which results from Bougerol's identity.

Proposition 3.2 *Consider* $(Z_t^\lambda, t \geq 0)$ *and* $(U_t^\lambda, t \geq 0)$ *two independent Ornstein-Uhlenbeck processes, the first one complex valued and the second one real valued, both starting from a point different from 0, and call* $T_b^{(\lambda)}(U^\lambda) = \inf\left\{t \geq 0 : e^{\lambda t} U_t^\lambda = b\right\}$. *Then, an Ornstein-Uhlenbeck extension of identity in law (ii) in Proposition 2.4 is the following:*

$$\theta^{Z^\lambda}_{T_b^{(\lambda)}(U^\lambda)} \stackrel{(law)}{=} C_{a(b)}, \tag{66}$$

where $a(x) = \arg\sinh(x)$.

Proof of Proposition 3.2 Let us consider a second Ornstein-Uhlenbeck process $(U_t^\lambda, t \geq 0)$ independent of the first one. Then, taking equation (60) for U_t^λ, we have:

$$e^{\lambda t} U_t^\lambda = \delta_{\left(\frac{e^{2\lambda t}-1}{2\lambda}\right)}, \tag{67}$$

where $(\delta_t, t \geq 0)$ is a complex valued Brownian motion starting from $z_0 = 1$. Thus:

$$T_b^{(\lambda)}(U^\lambda) = \frac{1}{2\lambda} \ln\left(1 + 2\lambda T_b^\delta\right). \tag{68}$$

Equation (61) for $t = \frac{1}{2\lambda} \ln\left(1 + 2\lambda T_b^\delta\right)$, equivalently: $\alpha(t) = T_b^\delta$ becomes (we suppose that $z_0 = 1$):

$$\theta^{Z^\lambda}_{T_b^{(\lambda)}(U^\lambda)} = \theta^{Z^\lambda}_{\frac{1}{2\lambda}\ln\left(1+2\lambda T_b^\delta\right)} = \theta^{\mathbb{B}}_{u=T_b^\delta} \stackrel{(law)}{=} C_{a(b)}.$$

□

3.2 On the distribution of $T^\theta_{-c,c}$ for an Ornstein-Uhlenbeck process

Now we turn to the study of the density function of:

$$T_c^{(\lambda)} \equiv T^{\theta^Z}_{-c,c} \equiv \inf\left\{t \geq 0 : \left|\theta_t^Z\right| = c\right\},$$

and its first moment.

Proposition 3.3 *Asymptotically for λ large, for $z_0 = 1$, we have:*

$$2\lambda\, E\left[T_c^{(\lambda)}\right] - \ln(2\lambda) \overset{\lambda\to\infty}{\longrightarrow} E\left[\ln\left(T^{\theta}_{-c,c}\right)\right], \tag{69}$$

and:

$$E\left[\ln\left(T^{\theta}_{-c,c}\right)\right] = 2\int_0^\infty \frac{dz}{\cosh\left(\frac{\pi z}{2}\right)} \ln\left(\sinh\left(cz\right)\right) + \ln\left(2\right) + c_E, \tag{70}$$

where c_E is Euler's constant.
For $c < \frac{\pi}{8}$, we have the asymptotic equivalence:

$$\frac{1}{\lambda}\left(E\left[T_c^{(\lambda)}\right] - E\left[\left(\sinh\left(B_{T^{\gamma}_{-c,c}}\right)\right)^2\right]\right) \overset{\lambda\to 0}{\longrightarrow} -\frac{1}{3}E\left[\left(\sinh\left(B_{T^{\gamma}_{-c,c}}\right)\right)^4\right]. \tag{71}$$

Equivalently:

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} E\left[T_c^{(\lambda)}\right] = \lim_{\lambda\to 0}\left[\frac{1}{\lambda}\left(E\left[T_c^{(\lambda)}\right] - E\left[T_c^{(0)}\right]\right)\right] = -\frac{1}{3}E\left[\left(\sinh\left(B_{T^{\gamma}_{-c,c}}\right)\right)^4\right]. \tag{72}$$

Moreover:

$$E\left[\left(\sinh\left(B_{T^{\gamma}_{-c,c}}\right)\right)^4\right] = \int_0^\infty \frac{dz}{\cosh\left(\frac{\pi z}{2}\right)}\left(\sinh\left(cz\right)\right)^4. \tag{73}$$

More precisely, for $c < \frac{\pi}{8}$:

$$E\left[\left(\sinh\left(B_{T^{\gamma}_{-c,c}}\right)\right)^4\right] = \frac{1}{8}\left(\frac{1}{\cos(4c)} - 4\frac{1}{\cos(2c)} + 3\right), \tag{74}$$

and asymptotically:

$$E\left[\left(\sinh\left(B_{T^{\gamma}_{-c,c}}\right)\right)^4\right] \underset{c\to 0}{\simeq} 5c^4. \tag{75}$$

Proof of Proposition 3.3
λ large
Let us return to equation (65). For $\lambda \to \infty$, we have:

$$\begin{aligned} E\left[T_c^{(\lambda)}\right] &= \frac{1}{2\lambda}E\left[\ln\left(1 + 2\lambda T^{\theta}_{-c,c}\right)\right] \\ &= \frac{1}{2\lambda}E\left[\ln\left(2\lambda\left(T^{\theta}_{-c,c} + \frac{1}{2\lambda}\right)\right)\right] \\ &= \frac{\ln(2\lambda)}{2\lambda} + \frac{1}{2\lambda}E\left[\ln\left(T^{\theta}_{-c,c} + \frac{1}{2\lambda}\right)\right]. \end{aligned}$$

Thus:

$$2\lambda\, E\left[T_c^{(\lambda)}\right] - \ln(2\lambda) \overset{\lambda\to\infty}{\longrightarrow} E\left[\ln\left(T^{\theta}_{-c,c}\right)\right],$$

which is precisely (69). Moreover, by the integral representation (51) for $E\left[\ln\left(T^{\theta}_{-c,c}\right)\right]$, we deduce (70).

<u>**λ small**</u>

We shall now study the case $\lambda \to 0$. We have that:

$$T_c^{(\lambda)} = \frac{1}{2\lambda}\ln\left(1 + 2\lambda T^{\theta}_{-c,c}\right).$$

For $c < \frac{\pi}{8}$, from Spitzer (3), (at least) the first two positive moments of $T^{\theta}_{-c,c}$ are finite: $E\left[\left(T^{\theta}_{-c,c}\right)^p\right] < \infty$, $(p = 1, 2)$. We make the elementary computation:

$$\begin{aligned}
\frac{1}{\lambda}\left(\frac{\ln(1+2\lambda x)}{2\lambda} - x\right) &= \frac{1}{\lambda}\left(\frac{1}{2\lambda}\int_1^{1+2\lambda x}\frac{dy}{y} - x\right)\\
&\overset{y=1+a}{=} \frac{1}{2\lambda^2}\int_0^{2\lambda x}\left(\frac{1}{1+a} - 1\right) da \overset{a=2\lambda b}{=} -2\int_0^{x}\frac{b\, db}{1+2\lambda b} \overset{\lambda\to 0}{\longrightarrow} -x^2.
\end{aligned}$$

Consequently, by replacing $x = T^{\theta}_{-c,c}$, we have:

$$\frac{1}{\lambda}\left(E\left[T_c^{(\lambda)}\right] - E\left[T^{\theta}_{-c,c}\right]\right) = E\left[-2\int_0^{T^{\theta}_{-c,c}}\frac{b\, db}{1+2\lambda b}\right].$$

We may now use the dominated convergence theorem [7], since the (db) integral is majorized by $(T^{\theta}_{-c,c})^2$, which is integrable. Thus:

$$\frac{1}{\lambda}\left(E\left[T_c^{(\lambda)}\right] - E\left[T^{\theta}_{-c,c}\right]\right) \overset{\lambda\to 0}{\longrightarrow} -E\left[(T^{\theta}_{-c,c})^2\right].$$

Following the proof of Proposition 2.10, Bougerol's identity and the scaling property yield:

$$\begin{aligned}
E\left[(\sinh(B_t))^2\right] &= E\left[(\delta_{A_t})^2\right] = E\left[A_t\,(\delta_1)^2\right] = E\left[A_t\right]\, E\left[(\delta_1)^2\right]\\
&= E\left[A_t\right].
\end{aligned}$$

Thus, for $t = T^{\gamma}_{-c,c}$, we have:

$$E\left[A_{T^{\gamma}_{-c,c}}\right] = E\left[\left(\sinh\left(B_{T^{\gamma}_{-c,c}}\right)\right)^2\right].$$

Similarly:

$$\begin{aligned} E\left[(\sinh(B_t))^4\right] &= E\left[(\delta_{A_t})^4\right] = E\left[(A_t)^2(\delta_1)^4\right] = E\left[(A_t)^2\right]\; E\left[(\delta_1)^4\right] \\ &= 3E\left[(A_t)^2\right]. \end{aligned}$$

Thus, for $t = T^{\gamma}_{-c,c}$, we have:

$$E\left[\left(A_{T^{\gamma}_{-c,c}}\right)^2\right] = \frac{1}{3}E\left[\left(\sinh\left(B_{T^{\gamma}_{-c,c}}\right)\right)^4\right].$$

So, because $A_{T^{\gamma}_{-c,c}} = T^{\theta}_{-c,c}$, we deduce (71). In order to prove (72), it suffices to remark that:

$$E\left[T^{(0)}_c\right] = E\left[T^{\theta}_{-c,c}\right] = E\left[A_{T^{\gamma}_{-c,c}}\right] = E\left[\left(\sinh\left(B_{T^{\gamma}_{-c,c}}\right)\right)^2\right].$$

On the one hand, by using the density of $B_{T^{\gamma}_{-c,c}}$:

$$\begin{aligned} E\left[\left(\sinh\left(B_{T^{\gamma}_{-c,c}}\right)\right)^4\right] &= \int_{-\infty}^{\infty}\frac{dy}{2c}\,\frac{1}{\cosh(\frac{\pi y}{2c})}\,(\sinh(y))^4 \\ &= \int_0^{\infty}\frac{dy}{c}\,\frac{1}{\cosh(\frac{\pi y}{2c})}\,(\sinh y)^4 \\ &\overset{z=\frac{y}{c}}{=} \int_0^{\infty} dz\,\frac{1}{\cosh(\frac{\pi z}{2})}\,(\sinh(cz))^4, \end{aligned}$$

which is finite if and only if $c < \frac{\pi}{8}$. In order to prove this, it suffices to use the standard expressions: $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^x + e^{-x}}{2}$. On the other hand (note $T \equiv T^{\gamma}_{-c,c}$), we remark that $-B_T \overset{(law)}{=} B_T$ and [32], ex.3.10, $E\left[e^{kB_T}\right] = E\left[e^{\frac{k^2}{2}T}\right] = \frac{1}{\cos(kc)}$, for $0 \le k < \pi(2c)^{-1}$, thus:

$$\begin{aligned} E\left[(\sinh(B_T))^4\right] &= \frac{1}{2^4}E\left[\left(e^{B_T} - e^{-B_T}\right)^4\right] \\ &= \frac{1}{2^4}E\left[e^{4B_T} - 4e^{3B_T - B_T} + 6e^{2B_T - 2B_T} - 4e^{B_T - 3B_T} + e^{-4B_T}\right] \\ &= \frac{1}{2^4}\left(2E\left[e^{4B_T}\right] - 8E\left[e^{2B_T}\right] + 6\right) \\ &= \frac{1}{2^3}\left(\frac{1}{\cos(4c)} - 4\frac{1}{\cos(2c)} + 3\right), \end{aligned}$$

which is precisely (74) and this is finite if and only if $c < \frac{\pi}{8}$. Moreover, asymptotically for $c \to 0$, by using the scaling property, we have:

$$\begin{aligned} E\left[\left(\sinh\left(B_{T^{\gamma}_{-c,c}}\right)\right)^4\right] &= E\left[\left(\sinh\left(cB_{T^{\gamma}_{-1,1}}\right)\right)^4\right] \overset{c\to 0}{\simeq} c^4 E\left[\left(B_{T^{\gamma}_{-1,1}}\right)^4\right] \\ &= c^4 3E\left[\left(T^{\gamma}_{-1,1}\right)^2\right] = 5c^4, \end{aligned}$$

since $E\left[\left(T^{\gamma}_{-1,1}\right)^2\right] = 5/3$ (see [31]; by using the notation of this paper, Table 3: $E\left[X_t^2\right] = \frac{t(2+3t)}{3}$ for $X_t = C_1$ and $t = 1$). This asymptotics may also be obtained by (74) by developing $\cos(4c)$ and $\cos(2c)$ into series up to the second order term and keeping the terms of the order c^4.

□

Remark 3.4 *If we slightly modify the above study for the Ornstein-Uhlenbeck process by inserting a diffusion coefficient D:*

$$Z_t = z_0 + \sqrt{2D}\tilde{Z}_t - \lambda \int_0^t Z_s ds\ ,$$

we obtain:

$$\begin{aligned} Z_t &= e^{-\lambda t}\left(z_0 + \sqrt{2D}\int_0^t e^{\lambda s} d\tilde{Z}_s\right) \\ &= e^{-\lambda t}\left(\mathbb{B}_{\alpha_t}\right), \end{aligned} \tag{76}$$

where in the second equation we used Dambis-Dubins-Schwarz Theorem with

$$\alpha_t = 2D\int_0^t e^{2\lambda s} ds = D\frac{e^{2\lambda t} - 1}{\lambda}$$

$$\Rightarrow \alpha_t^{-1} = \frac{1}{2\lambda}\ln\left(1 + \frac{\lambda}{D}t\right).$$

Thus:

$$2\lambda\, E\left[T_c^{(\lambda)}\right] - \ln\left(\frac{\lambda}{D}\right) \overset{\lambda\to\infty}{\longrightarrow} E\left[\ln\left(T^{\theta}_{-c,c}\right)\right], \tag{77}$$

because:

$$\begin{aligned}
E\left[T_c^{(\lambda)}\right] &= \frac{1}{2\lambda} E\left[\ln\left(1+\frac{\lambda}{D}T^{\theta}_{-c,c}\right)\right] \\
&= \frac{1}{2\lambda} E\left[\ln\left(\frac{\lambda}{D}\left(T^{\theta}_{-c,c}+\frac{D}{\lambda}\right)\right)\right] \\
&= \frac{\ln\left(\frac{\lambda}{D}\right)}{2\lambda} + \frac{1}{2\lambda} E\left[\ln\left(T^{\theta}_{-c,c}+\frac{D}{\lambda}\right)\right].
\end{aligned}$$

Moreover:

$$\begin{aligned}
E\left[\ln\left(T^{\theta}_{-c,c}\right)\right] &= 2\ln(z_0) + E\left[\ln\left(T^{\theta(1)}_{-c,c}\right)\right] \\
&= 2\ln(z_0) + \int_0^{\infty} \frac{dz}{\cosh\left(\frac{\pi z}{2}\right)} \ln\left(\sinh\left(cz\right)\right) + \ln\left(2\right) + c_E,
\end{aligned} \tag{78}$$

where $T^{\theta(1)}_{-c,c}$ denotes the first hitting time of the symmetric conic boundary of angle c for a Brownian motion Z starting from 1.
For λ small, we replace $2T^{\theta}_{-c,c}$ by $\frac{z_0^2}{D}T^{\theta}_{-c,c}$ in the proof of Proposition 3.3 (λ small case) and we have:

$$T_c^{(\lambda)} = \frac{1}{2\lambda}\ln\left(1+\lambda\frac{z_0^2}{D}T^{\theta}_{-c,c}\right).$$

By repeating the previous calculation, we make the elementary computation:

$$\frac{1}{\lambda}\left(\frac{\ln\left(1+\frac{z_0^2}{D}x\right)}{2\lambda} - \frac{z_0^2}{D}x\right) = -\frac{1}{2}\int_0^x \frac{\left(\frac{z_0^2}{D}\right)^2 b\, db}{1+\lambda\frac{z_0^2}{D}b} \overset{\lambda\to 0}{\longrightarrow} -\left(\frac{z_0^2}{2D}\right)^2 x^2.$$

We replace $x = T^{\theta}_{-c,c}$, and by the dominated convergence theorem [7], for $c < \frac{\pi}{8}$, we obtain:

$$\begin{aligned}
\frac{1}{\lambda}\left(E\left[T_c^{(\lambda)}\right] - \frac{z_0^2}{2D}E\left[\left(\sinh\left(B_{T^{\gamma}_{-c,c}}\right)\right)^2\right]\right) &\overset{\lambda\to 0}{\longrightarrow} -\frac{1}{3}\left(\frac{z_0^2}{2D}\right)^2 E\left[(T^{\theta}_{-c,c})^2\right] \\
&= -\frac{1}{3}\left(\frac{z_0^2}{2D}\right)^2 E\left[\left(\sinh\left(B_{T^{\gamma}_{-c,c}}\right)\right)^4\right],
\end{aligned}$$

where $E\left[\left(\sinh\left(B_{T^{\gamma}_{-c,c}}\right)\right)^4\right]$ is given by (73), (74) and asymptotically, for $c \to 0$ by (75).

Acknowledgements
I would like to thank Alain Comtet for his contribution in the proof of Proposition 2.8 and the Human Frontier Science Program for its support. This article represents the first part of my PhD thesis, under the supervision of Professors D. Holcman and M. Yor.

References

[1] M. Abramowitz and I.A. Stegun (1970). Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables. Dover Publications, Inc., New York.

[2] L. Alili, D. Dufresne, and M. Yor (1997). Sur l'identité de Bougerol pour les fonctionnelles exponentielles du mouvement Brownien avec drift. In *Exponential Functionals and Principal Values related to Brownian Motion. A collection of research papers; Biblioteca de la Revista Matematica, Ibero-Americana*, ed. M. Yor, 3-14.

[3] D. André (1887). Solution directe du problème résolu par M. Bertrand. *C. R. Acad. Sci. Paris*, **105**, 436-437.

[4] V. Bentkus, G. Pap, and M. Yor (2003). Optimal bounds for Cauchy approximations for the winding distribution of planar Brownian motion. *J. Theor. Probab.,* **16** (2), ISSN 0894-9840, p. 259-279.

[5] J. Bertoin and W. Werner (1994). Asymptotic windings of planar Brownian motion revisited via the Ornstein-Uhlenbeck process. *Sém. Prob. XXVIII, Lect. Notes in Mathematics,* **1583**, Springer, Berlin Heidelberg New York 138-152.

[6] P. Biane and M. Yor (1987). Valeurs principales associées aux temps locaux browniens. *Bull. Sci. Math.*, **111**, 23-101.

[7] P. Billingsley (1995). Probability and Measure. 3rd ed., John Wiley and Sons, New York.

[8] Ph. Bougerol (1983). Exemples de théorèmes locaux sur les groupes résolubles. *Ann. Inst. H. Poincaré*, **19**, 369-391.

[9] D. Burkholder (1977). Exit times of Brownian Motion, Harmonic Majorization and Hardy Spaces. *Adv. in Math.*, **26**, 182-205.

[10] L. Chaumont and M. Yor (2003). Exercises in Probability: A Guided Tour from Measure Theory to Random Processes, via Conditioning. Cambridge University Press.

[11] A. Comtet (2006). Personal Communication.

[12] A. Comtet, C. Monthus, and M. Yor (1998). Exponential functionals of Brownian motion and disordered systems. *J. Appl. Prob.*, **35**, 255-271.

[13] C. Donati-Martin, H. Matsumoto, and M. Yor (2000). On positive and negative moments of geometric Brownian motion. *Statistics and Probability Letters*, **49**, 45-52.

[14] D. Dufresne (2000). Laguerre Series for Asian and Other Options. *Mathematical Finance*, Vol. **10**, No. 4, 407-428.

[15] D. Dufresne and M. Yor (2010). A two-dimensional extension of Bougerol's identity in law for the exponential functional of Brownian motion: the story so far... Preprint.

[16] R. Durrett (1982). A new proof of Spitzer's result on the winding of 2-dimensional Brownian motion. *Ann. Prob.* **10**, 244-246.

[17] L. Gallardo (2008). Mouvement Brownien et calcul d'Itô. Hermann.

[18] I.S. Gradshteyn and I.M. Ryzhik (1965). Table of Integrals, Series and Products. Academic Press.

[19] K. Ishiyama (2005). Methods for Evaluating Density Functions of Exponential Functionals Represented as Integrals of Geometric Brownian Motion. *Methodology and Computing in Applied Probability*, Springer **7**, 271-283.

[20] K. Itô and H.P. McKean (1965). Diffusion Processes and their Sample Paths. Springer, Berlin Heidelberg New York. New edition in: Classics in Mathematics (1996).

[21] J.F. Le Gall (1990-1992). Some properties of planar Brownian motion. *Ecole d'été de Saint-Flour XX*, 1990, *Lect. Notes in Mathematics*, Springer, Berlin Heidelberg New York 1992, **1527**, 112-234.

[22] J.F. Le Gall and M. Yor (1987). Etude asymptotique des enlacements du mouvement brownien autour des droites de l'espace. *Prob. Th. Rel. Fields* **74**, 617-635.

[23] N.N. Lebedev (1972). Special Functions and their Applications. Dover Publications, Inc., New York.

[24] D. Dugué (1980). Œuvres de Paul Lévy, Vol. IV, Processus Stochastiques, Gauthier-Villars. **158** Random Functions: General Theory with Special Reference to Laplacian Random Functions by Paul Lévy.

[25] R. Mansuy and M. Yor (2008). Aspects of Brownian Motion. Springer-Verlag, Berlin Heidelberg.

[26] H. Matsumoto and M. Yor (1998). On Bougerol and Dufresne's identities for exponential Brownian functionals. *Proc. Japan Acad. Ser. A Math. Sci.*, Volume **74**, Number 10, 152-155.

[27] H. Matsumoto and M. Yor (2005). Exponential functionals of Brownian motion, I: Probability laws at fixed time.*Probab. Surveys* Volume **2**, 312-347.

[28] P. Messulam and M. Yor (1982). On D. Williams' "pinching method" and some applications. *J. London Math. Soc.*, **26**, 348-364.

[29] G. Pap and Yor, M. (2000). The accuracy of Cauchy approximation for the windings of planar Brownian motion. *Periodica Mathematica Hungarica* **41**, 213-226.

[30] J.W. Pitman and M. Yor (1986). Asymptotic Laws of planar Brownian Motion. *Ann. Prob.* **14**, 733-779.

[31] J.W. Pitman and M. Yor (2003). Infinitely divisible laws associated with hyperbolic functions. *Canad. J. Math.* **55**, 292-330.

[32] D. Revuz and M. Yor (1999). Continuous Martingales and Brownian Motion. 3rd ed., Springer, Berlin.

[33] F. Spitzer (1958). Some theorems concerning two-dimensional Brownian Motion. *Trans. Amer. Math. Soc.* **87**, 187-197.

[34] P. Vallois (1992). Amplitude du mouvement Brownien et juxtaposition des excursions positives et ngatives. *Sém. Prob. XXVI, Lect. Notes in Mathematics,* Springer-Verlag, p. 361-373.

[35] D. Williams (1974). A simple geometric proof of Spitzer's winding number formula for 2-dimensional Brownian motion. University College, Swansea. Unpublished.

[36] M. Yor (1980). Loi de l'indice du lacet Brownien et Distribution de Hartman-Watson. *Z. Wahrsch. verw. Gebiete*, **53**, 71-95.

[37] M. Yor (1985). Une décomposition asymptotique du nombre de tours du mouvement brownien complexe. [An asymptotic decomposition of the winding number of complex Brownian motion]. *Colloquium in honor of Laurent Schwartz*, Vol. **2** (Palaiseau, 1983). *Astérisque* No. **132** (1985), 103–126.

[38] M. Yor (1997). Generalized meanders as limits of weighted Bessel processes, and an elementary proof of Spitzer's asymptotic result on Brownian windings. *Studia Scient. Math. Hung.* **33**, 339-343.

[39] M. Yor (2001). Exponential Functionals of Brownian Motion and Related Processes. Springer Finance. Springer-Verlag, Berlin.